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by Jerrold H. Suddath and Waldo I. Oehman

Langley Research Center

Langley Station, Hampton, Va.

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SUMMARY

Newtonian flow theory and the calculus of variations were used to study minimum drag shapes for bodies with cross-sectional ellipticity at hypersonic speeds and zero angle of attack. The study was made for conditions of given length and base height, given length and volume, given base height and volume, and given base height and surface area. Numerical examples for conditions of given length and base height and given length and volume are presented to assess the effect of cross-sectional ellipticity on body shape and pressure drag coefficient. The method of steepest descent was used for computations of body shapes with given length and volume.

INTRODUCTION

The problem of determining the shapes of bodies of revolution with minimum pressure drag at hypersonic speeds was treated in reference 1. The analysis therein was based on Newtonian flow theory and the calculus of variations, and treated a variety of constraints comprising combinations of length, volume, base diameter, and surface area. If bodies with elliptical rather than circular cross sections are considered, the solution will no doubt be lengthier, but of similar form. As yet, minimum drag bodies with elliptic cross sections have not been treated; however, experiments with bodies with elliptical sections have shown (refs. 2 and 3) that increasing cross-sectional ellipticity (that is, increasing the ratio of horizontal axis length to vertical axis length) results in higher lift-drag ratio at both subsonic and supersonic speeds. Therefore, the theoretical determination of minimum drag shapes with elliptic cross sections would provide the bases for extrap[illegible] of present data and for further experimental investigations.

This report presents some theoretical results concerning the effect of cross-sectional ellipticity on the geometry and pressure drag of minimum drag bodies. The calculus of variations will be applied to obtain qualitative results for conditions of given base height and volume and given base height and surface area; whereas, a more detailed study will be applied to the conditions of given base height and length, and given length and volume. Quantitative results for given length and volume will also be obtained by the method of steepest descent. This numerical method, developed by Kelley et al. (ref. 4), is an iterative procedure that may be programed for a digital computer.

SYMBOLS

A area of body cross section

C_D pressure drag coefficient

$C, C_1, C_2, C_3, C_4, C_5$ constants of integration

c constant

D drag, lb

$E\left(\frac{\pi}{2}, k\right)$ complete elliptic integral of second kind

$F_y = \frac{\partial F}{\partial y}$

$F_{y'} = \frac{\partial F}{\partial y'}$

$F_{y'y'} = \frac{\partial^2 F}{\partial y'^2}$

Delete symbol $[F]_y$ and the definition for it.

$F(y,y')$ integrand of equation (5)

$f(y,y')$ integrand function (see eq. (3))

G integral defined by equation (2)

$g(y,y')$ integrand of equation (2)

I integral defined by equation (3)

J integral defined by equation (5)

δJ first variation of J

$K\left(\frac{\pi}{2}, k\right)$ complete elliptic integral of first kind

k modulus of elliptic integrals $\left(k^2 = \frac{1 - \mu^2}{1 + y'^2}\right)$

l body length, ft

m constant

n fineness ratio, $\frac{\sqrt{\mu}l}{2y(l)}$

p exponent for power-law body shape, $\left(\eta = m\xi^p\right)$

q_∞ free-stream dynamic pressure, lb/sq ft

S surface area, sq ft

V volume, cu ft

x,y,z orthogonal coordinates of body

Page 3, line 8: Delete symbol δy and the definition for it.

Page 3, line 8 and following lines: Insert the following symbols and definitions:

δ denotes variation consistent with prescribed boundary conditions

$\tilde{\delta}$ denotes variation taken at a constant station x

$\alpha(y_o)$ arbitrary prescribed function of y_o

ξ nondimensional coordinate x/l

Subscripts:

o condition at body nose

l condition at body base

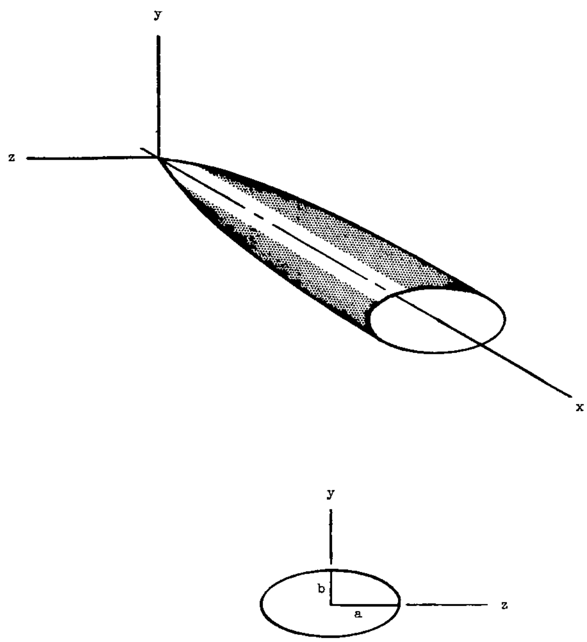
max maximum

A prime (') denotes differentiation with respect to the independent variable.

PROBLEM FORMULATION AND SOLUTION

Statement of the Problem

The problem considered herein is stated as follows: Given a body with cross-sectional ellipticity, find the shape that minimizes the pressure drag and satisfies (in some instances) an integral constraint. The body is assumed to be at zero angle of attack and the pressure drag is obtained from Newtonian flow theory.



Typical body section: ellipticity, $\mu = \frac{b}{a}$

Figure 1.- Sketch showing coordinate system.

The geometry of the problem is represented by figure 1. The body is assumed to be moving in the negative x-direction. Any plane given by the equation $x = c$, with $0 \leqq c \leqq l$ intersects the body in an ellipse with minor and major axes parallel to the y and z axes, respectively. For every value of x in the interval $0 \leqq x \leqq l$, the ratio of the minor to the major axis of the ellipse is the constant μ.

The Newtonian flow theory (see ref. 5) shows that the pressure drag acting on the body may be given by

$$D = \frac{2\pi q_\infty}{\mu}\left[y_o^2 + \int_0^l \frac{2yy'^3dx}{\sqrt{(1 + y'^2)(\mu^2 + y'^2)}}\right] \qquad (1)$$

where $y' = \frac{dy}{dx}$ is the slope of the body in the x,y plane. Mathematically, the problem reduces to the x,y plane because of symmetry and constant μ.

The integral constraint, when required, will be denoted by

$$G = \alpha(y_o) + \int_0^l g(y,y')dx \qquad (2)$$

The problem may now be stated mathematically as follows: Find the function $y = y(x)$ so that D is minimized and subject to the constraint that G be a prescribed value.

Method of Solution and Resulting Equations

The problem may be conveniently solved by the method of the calculus of variations. In applying the calculus of variations (ref. 6), it is convenient to define the quantity I as

$$I \equiv \frac{\mu D}{2\pi q_\infty} = y_o^2 + \int_0^{x_1} f(y,y')dx \qquad (3)$$

where the upper limit of integration x_1 is variable to permit variations in the body length and

$$f(y,y') \equiv \frac{2yy'^3}{\sqrt{(1+y'^2)(\mu^2+y'^2)}} \tag{4}$$

then J, the quantity to be minimized, is given by

$$J = I + \lambda G = H(y_o) + \int_0^{x_1} F(y,y')dx \tag{5}$$

where

$$F(y,y') \equiv f(y,y') + \lambda g(y,y') \quad \text{and} \quad H(y_o) \equiv y_o^2 + \lambda\alpha(y_o) \tag{6}$$

and λ is a constant Lagrange multiplier. Calculating the first variation of J and setting it equal to zero leads to

$$\delta J = H_{y_o}\delta y_o + \left[(F - y'F_{y'})\delta x + F_{y'}\delta y\right]\Big|_{x=0}^{x=x_1} + \int_0^{x_1}\left(F_y - \frac{d}{dx}F_{y'}\right)\tilde{\delta}y\ dx = 0 \tag{7}$$

From familiar arguments concerning the arbitrary variation δy, equation (7) leads to the following four conditions:

(1) The Euler-Lagrange equation which must be satisfied over the interval $0 \leqq x \leqq x_1$ is

$$F_y - \frac{d}{dx}F_{y'} = 0 \tag{8a}$$

Since x does not appear explicitly in $F(y,y')$, the first integral of equation (8a) is

$$y'F_{y'} - F(y,y') = C \tag{8b}$$

where C is a constant.

(2) The terminal condition which must be satisfied at $x = 0$ is

$$H_{y_o} - \left(F_{y'}\right)_{x=0} = 0 \tag{9}$$

(3) The terminal condition which must be satisfied at $x = x_1$ is either

$$F_{y'}\Big|_{x=x_1} = 0 \tag{10}$$

"when the base height is not prescribed, or"

$$\left(y'F_{y'} - F(y,y')\right)\Big|_{x=x_1} = 0 \tag{11}$$

when the body length is not specified.

(4) Furthermore, the solution $y(x)$ must be such that the Legendre condition is satisfied everywhere along the extremal; that is

$$F_{y'y'} \geqq 0 \tag{12}$$

It will be seen, a posteriori, that it is unnecessary to consider the Weierstrass-Erdmann vertex condition.

Specific Solutions

From the preceding discussion, it is apparent that some information about the integral constraint and the terminal conditions must be specified in order to obtain a meaningful solution to the problem. Therefore, the solution curve $y(x)$ will be examined analytically for conditions of given length and base height, given length and volume, given base height and volume, and given base height and surface area. Integral constraints arise when either the volume or the surface area is specified and are

$$G = \frac{\mu V}{\pi} = \int_0^{x_1} y^2 dx \tag{13}$$

for given volume, and

$$F = \frac{\mu S}{2} = \frac{\pi y_o^2}{2} + \int_0^{x_1} 2y\sqrt{1 + y'^2}\,E\left(\frac{\pi}{2}, k\right)dx \tag{14}$$

for given surface area where $E\left(\frac{\pi}{2}, k\right)$ is a complete elliptic integral of the second kind and

$$k^2 = \frac{1 - \mu^2}{1 + y'^2}$$

Specified base height and length.- The function $F(y,y')$ given by equation (6) for the problem of specified base height and length, in explicit form, is

$$F(y,y') = \frac{2yy'^3}{\sqrt{\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)}} \tag{15}$$

Equations (8b) to (12) may be written explicitly in the following way:

(1) The first integral of the Euler-Lagrange equation, equation (8b), is

$$\frac{2yy'^3\left[2\mu^2 + \left(1 + \mu^2\right)y'^2\right]}{\left[\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)\right]^{3/2}} = C_1 \tag{16}$$

(2) The terminal condition at $x = 0$, equation (9), is

$$2y\left\{\frac{y'^2\left(3\mu^2 + 2\left(1 + \mu^2\right)y'^2 + y'^4\right)}{\left[\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)\right]^{3/2}} - 1\right\}\Bigg|_{x=0} = 0 \tag{17}$$

(3) For the terminal condition at the base of the body neither equation (10) nor (11) apply since length and base height are given.

(4) The Legendre condition (inequality (12)) in explicit form can be written as

$$yy'\left[6\mu^4 + 5\mu^2\left(1 + \mu^2\right)y'^2 + 2\left(1 - \mu^2 + \mu^4\right)y'^4 - \left(1 + \mu^2\right)y'^6\right] \geqq 0 \tag{18}$$

From simple physical considerations, the solution $y(x)$ should satisfy $y(x) \geqq 0$ for $0 \leqq x \leqq l$, and, it will be assumed that $y(l) > 0$. With $y(x) > 0$ the Legendre condition is satisfied if

$$0 \leqq y'(x) \leqq y'_{max}(\mu)$$

The curve $y'_{max}(\mu)$ is plotted in figure 2 and is seen to agree with the well-known result (ref. 5) that $y'_{max}(l) = \sqrt{3}$ for the body of revolution.

Equation (16) requires that $y(0) > 0$. If $y(0) = 0$, then $C_1 = 0$ and this condition requires either that $y(x) = 0$ or that $y'(x) = 0$. Physical conditions, however, make $y(x) = 0$ untenable, and terminal conditions at $x = 0$ require that $y'(x) > 0$. Therefore, $y(0)$ must be greater than zero; and, thus, the body has a flat nose.

With $y(0) > 0$, equation (17) yields $y'(0)$ as a function of the ellipticity parameter μ. For convenience, this function is plotted in figure 2 and is well below the Legendre boundary.

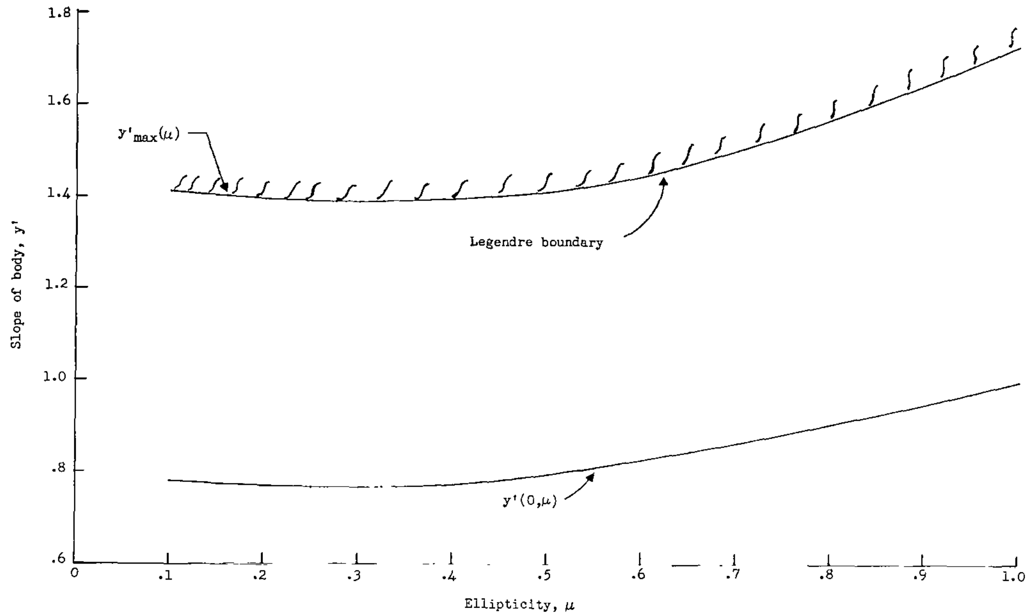


Figure 2.- Effect of ellipticity on slope of minimum drag bodies.

Equation (16) may be solved for $y(x)$ to obtain

$$y = \frac{C_1\left[\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)\right]^{3/2}}{2y'^3\left[2\mu^2 + \left(1 + \mu^2\right)y'^2\right]} \tag{19}$$

Now

$$x = \int_{y(0)}^{y(x)} \frac{dy}{y'} = \int_{y'(0)}^{y'(x)} \frac{1}{y'} \frac{dy}{dy'} \, dy' \tag{20}$$

Taking the derivative of equation (19) with respect to y', substituting the result into equation (20), and subsequently integrating yields

$$\begin{aligned} x = \frac{C_1}{4}\Bigg(\Bigg\{&\frac{3\left[2\mu^2 + \left(1 + \mu^2\right)y'^2\right]}{8\mu^2 y'^4} - \frac{\left(1 - \mu^2\right)^2}{2\mu^2\left[2\mu^2 + \left(1 + \mu^2\right)y'^2\right]} - \frac{\left(1 + \mu^2\right)}{4\mu^2 y'^2}\Bigg\}\left[\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)\right]^{1/2} \\ &+ \frac{1 - 10\mu^2 + \mu^4}{16\mu^3} \log_e\left[\frac{\sqrt{\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)} + \mu}{y'^2} + \frac{1 + \mu^2}{2\mu}\right] \\ &+ \frac{1}{1 + \mu^2} \log_e\left[\sqrt{\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)} + y'^2 + \frac{1 + \mu^2}{2}\right] - \frac{\left(1 - \mu^2\right)^3}{8\mu^3\left(1 + \mu^2\right)}\left[\sin^{-1} \frac{\left(1 - \mu^2\right)y'^2}{2\mu^2 + \left(1 + \mu^2\right)y'^2}\right]\Bigg) + C_2 \end{aligned} \tag{21}$$

The constant of integration C_2 in equation (21) may be determined in terms of C_1 by substituting $y'(0)$ and $x = 0$ for y' and x, respectively. Since $y(l)$ and $x = l$ are given, they may be substituted into equations (19) and (21) to obtain two equations which may be solved simultaneously to determine $y'(l)$ and C_1. Finally, by choosing values of y' between $y'(0)$ and $y'(l)$, x and $y(x)$ may be computed to define the body shape completely. The numerical work may be programed for a digital computer.

Specified length and volume.- The integral constraint for the problem of specified length and volume is given by equation (13) and the integrand function is

$$F(y,y') = \frac{2yy'^3}{\sqrt{\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)}} + \lambda y^2 \tag{22}$$

Equations (8b) to (12) in explicit form are as follows:

(1) The first integral of the Euler-Lagrange equation is

$$\frac{2yy'^3\left[2\mu^2 + \left(1 + \mu^2\right)y'^2\right]}{\left[\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)\right]^{3/2}} - \lambda y^2 = C_3 \tag{23}$$

(2) The terminal condition at $x = 0$ is the same as for the preceding problem. Therefore, equation (17) is applicable.

(3) The terminal condition at $x = l$ (eq. (10)) is

$$\left.\frac{2yy'^2\left[3\mu^2 + 2\left(1 + \mu^2\right)y'^2 + y'^4\right]}{\left[\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)\right]^{3/2}}\right|_{x=l} = 0 \tag{24}$$

(4) The discussion of the Legendre condition presented for the preceding problem is applicable for the present problem.

For $y(l) > 0$, equation (24) requires that $y'(l) = 0$ and thus $C_3 = -\lambda\left[y(l)\right]^2$. (See eq. (23).) Furthermore, equation (23) yields

$$\lambda y = \alpha(\mu,y') - \sqrt{\alpha^2(\mu,y') + \left(\lambda y_1\right)^2} \tag{25}$$

where

$$\alpha(\mu,y') \equiv \frac{y'^3\left[2\mu^2 + \left(1 + \mu^2\right)y'^2\right]}{\left[\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)\right]^{3/2}} \tag{26}$$

from which it can be seen that $y(0) \neq 0$. This nonidentity follows from the fact that $y_1 > 0$, and the right-hand side of equation (25) cannot be identically zero if $y_1 \neq 0$.

Now,

$$\lambda x = \int_{y(0)}^{y(x)} \frac{d(\lambda y)}{y'} = \int_{y'(0)}^{y'(x)} \frac{1}{y'}\frac{d(\lambda y)}{dy'}\,dy' = \int_{y'(0)}^{y'(x)}\left[1 - \frac{\alpha(\mu,y')}{\sqrt{\alpha^2(\mu,y') + (\lambda y_1)^2}}\right]\eta(\mu,y')dy' \tag{27}$$

where

$$\eta(\mu,y') \equiv \frac{y'\left[6\mu^4 + 5\mu^2\left(1 + \mu^2\right)y'^2 + 2\left(1 - \mu^2 + \mu^4\right)y'^4 - \left(1 + \mu^2\right)y'^6\right]}{\left[\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)\right]^{5/2}}$$

Further, the volume integral is

$$\lambda^3 V = \frac{\pi}{\mu}\int_0^{l}(\lambda y)^2 d(\lambda x) = \frac{\pi}{\mu}\int_{y'(0)}^{0}\left[\alpha - \sqrt{\alpha^2 + \left(\lambda y_1\right)^2}\right]^2 \eta\left[1 - \frac{\alpha}{\sqrt{\alpha^2 + \left(\lambda y_1\right)^2}}\right]dy' \tag{28}$$

Numerical solutions may be obtained by the following procedure:

(1) Approximate a value for λy_1 and substitute it into equation (25) to obtain λy as a function of μ and y'.

(2) Integrate equation (27) from $y'(0)$ to $y'(l) = 0$ to evaluate λl and, hence, λ.

(3) The value of λy_1 is adjusted until the computed value of λ satisfies the volume constraint (eq. (28)) for the specified volume.

(4) The body shape may then be computed by simultaneous solution of equations (25) and (27).

In addition to this procedure, the method of steepest descent, which is a powerful tool for computing optimum trajectories, may also be used to compute the desired minimum drag body shapes. This method was in fact used to make the calculation in preference to the previously outlined procedure because of the convenience it afforded the authors. The appendix presents the necessary reformulation of the problem so that the method of steepest descent, which is discussed at length, may be used to obtain solutions.

Specified base height and volume.- The integrand function for specified base height and volume is the same as for the preceding problem.

$$F(y,y') = \frac{2yy'^3}{\sqrt{(1 + y'^2)(\mu^2 + y'^2)}} + \lambda y^2 \tag{29}$$

The other equations of importance are as follows:

(1) The first integral of the Euler-Lagrange equation is

$$2y\beta(\mu,y') - \lambda y^2 = C_4 \tag{30}$$

where

$$\beta(\mu,y) = \frac{y'^3\left[2\mu^2 + (1 + \mu^2)y'^2\right]}{\left[(1 + y'^2)(\mu^2 + y'^2)\right]^{3/2}} \tag{31}$$

(2) Equation (17) is applicable for the terminal condition at $x = 0$, and $y'(0)$ is a function of μ. (See fig. 2.)

(3) The terminal condition at $x = x_1$ (eq. (11)) is

$$\left(2y\beta(\mu,y') - \lambda y^2\right)\Big|_{x=x_1} = 0 \tag{32}$$

(4) The Legendre condition given by equation (18) and the discussion of equation (18) apply to the present problem.

Equation (32) requires that C_4 (eq. (30)) be zero. Thus, equation (30) may be solved for $y = y(y',\lambda)$ to give

$$y = \frac{2\beta(\mu,y')}{\lambda} \tag{33}$$

The condition that $y(0) = 0$ satisfies the terminal constaint at the body nose (eq. (17)), and from equation (33), the slope at the nose is zero

$(y'(0) = 0)$. Further, the slope of the body must increase monotonically as x increases. However, the slope must not exceed the values given by the Legendre boundary (fig. 2). Thus, the bodies for a given base height and volume will have a cusped shape.

The minimum drag body shape may now be completely defined by equation (33) and

$$x = \int_{y(0)}^{y(x)} \frac{dy}{y'} = \int_{y'(0)=0}^{y'(x)} \frac{1}{y'} \frac{dy}{dy'} dy' = \int_{y'(0)=0}^{y'(x)} \frac{2}{\lambda y'} \frac{d\beta}{dy'} dy' \tag{34}$$

which integrates to give

$$x = \frac{2}{\lambda}\left\{\frac{\left(1 + y'^2\right)\left[3\mu^2 y'^2 + \left(2 - \mu^2\right)y'^4\right] - \left(1 - \mu^2\right)y'^6}{\left[\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)\right]^{3/2}} + \frac{\left(\mu^8 - \mu^4 + 8\mu^2 + 2\right)\left[\mu\left(2y'^2 + 1 + \mu^2\right) - \left(1 + \mu^2\right)\sqrt{\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)}\right]}{2\mu\left(1 - \mu^2\right)^4\sqrt{\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)}}\right\} \tag{35}$$

for $0 < \mu < 1$ and

$$x = \frac{2}{\lambda}\left[\frac{y'^4 + 3y'^2}{\left(1 + y'^2\right)^2}\right] \tag{36}$$

for $\mu = 1$ (body of revolution).

In order to avoid unnecessary computations that result from some selections of volume and base height, a useful guide is obtained by considering the volume integral given by

$$V = \frac{\pi}{\mu}\int_0^{x_1} y^2 dx = \frac{\pi}{\mu}\int_0^{y_1'} \frac{y^2}{y'} \frac{dy}{dy'} dy' \tag{37}$$

or, explicitly,

$$\frac{V}{y_1^3} = \frac{8\pi}{\mu\left(\lambda y_1\right)^3}\int_0^{y_1'} \frac{y'^7\left[2\mu^2 + \left(1 + \mu^2\right)y'^2\right]\left[6\mu^4 + 5\mu^2\left(1 + \mu^2\right)y'^2 + 2\left(1 - \mu^2 + \mu^4\right)y'^4 - \left(1 + \mu^2\right)y'^6\right]}{\left[\left(1 + y'^2\right)\left(\mu^2 + y'^2\right)\right]^{11/2}} dy' \tag{38}$$

where

$$\lambda y_1 = \frac{2y_1'^3\left[2\mu^2 + \left(1 + \mu^2\right)y_1'^2\right]}{\left[\left(1 + y_1'^2\right)\left(\mu^2 + y_1'^2\right)\right]^{3/2}} \tag{39}$$

The minimum allowable value of the ratio V/y_1^3 may be obtained by integrating equation (38) with the slope given by the Legendre boundary $\left(y_1' = y'_{max}(\mu)\right)$ as the upper limit of integration. A plot of $\frac{V\left(y_1' = y'_{max}(\mu)\right)}{y_1^3}$ is presented as a function of ellipticity μ in figure 3.

Minimum drag body shapes for given base height and volume (with $y_o = y_o' = 0$) may be computed by the following procedure. With the base height and volume chosen such that

$$\frac{V}{y_1^3} \geqq \frac{V\left(y_1' = y'_{max}(\mu)\right)}{y_1^3} \quad \text{(fig. 3)},$$

equation (33) and either equation (35) or (36) are solved simultaneously for y and x for an assumed value of λ. The value of λ is adjusted until the resulting body shape satisfies the integral constraint (eq. (13)) for the desired volume.

Bodies having blunt noses $(y(0) > 0)$ are also possible, but generally they have a fineness ratio much less than 1 and the height at the nose is of the same order of magnitude as the base height. Consequently, the bodies resemble thin disks with cusped edges. Equations (33) and (34) must be solved simultaneously to obtain the body shape, whereas λ and y_o must be adjusted to satisfy the volume constraint and the terminal conditions.

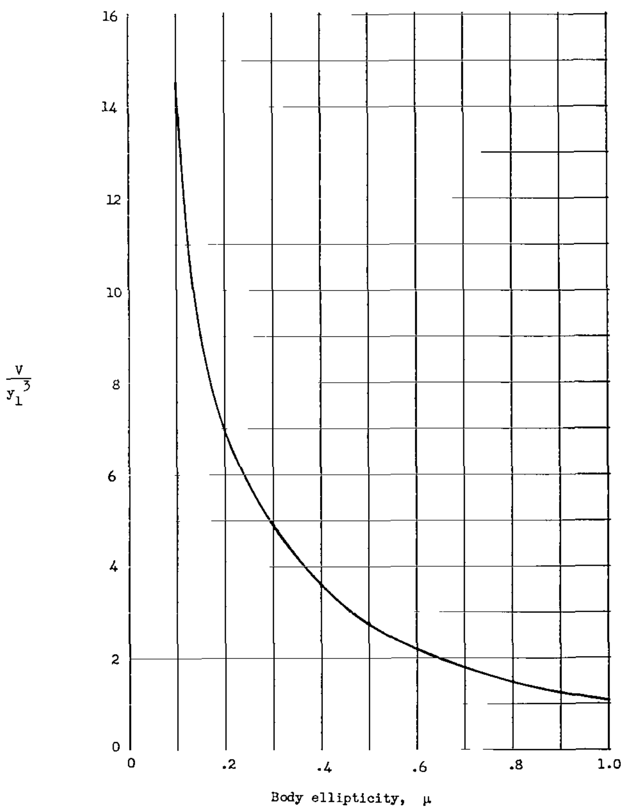


Figure 3.- Minimum values of V/y_1^3 for minimum drag body shapes with given base height and volume. $y_1' = y'_{max}(\mu)$.

<u>Specified base height and surface area</u>.- The integral constraint for the problem of specified base height and surface area is given by equation (14), and the integrand function is

$$F(y,y') = 2y\left[\frac{y'^3}{\sqrt{(1 + y'^2)(\mu^2 + y'^2)}} + \lambda\sqrt{1 + y'^2}E\left(\frac{\pi}{2},k\right)\right] \tag{40}$$

Equations (8b) to (12) then are as follows:

(1) The first integral of the Euler-Lagrange equation is

$$2y\left(\frac{y'^3\left[2\mu^2 + (1 + \mu^2)y'^2\right]}{\left[(1 + y'^2)(\mu^2 + y'^2)\right]^{3/2}} + \lambda\left\{\frac{y'^2\left[K\left(\frac{\pi}{2},k\right) - E\left(\frac{\pi}{2},k\right)\right] - E\left(\frac{\pi}{2},k\right)}{\sqrt{1 + y'^2}}\right\}\right) = C_5 \tag{41}$$

where $K\left(\frac{\pi}{2},k\right)$ and $E\left(\frac{\pi}{2},k\right)$ are complete elliptic integrals of the first and second kind, respectively.

(2) The terminal condition at $x = 0$ is

$$2y(0)\left\{\frac{y'^2\left[3\mu^2 + 2(1 + \mu^2)y'^2 + y'^4\right]}{\left[(1 + y'^2)(\mu^2 + y'^2)\right]^{3/2}} + \frac{\lambda y'K\left(\frac{\pi}{2},k\right)}{\sqrt{1 + y'^2}} - \frac{\pi\lambda}{2} - 1\right\}\Bigg|_{x=0} = 0 \tag{42}$$

(3) The terminal condition at $x = x_1$ (eq. (11)) requires that $C_5 = 0$ (eq. (41)).

(4) The Legendre condition in explicit form is

$$F_{y'y'} = \frac{2yy'}{\left[(1 + y'^2)(\mu^2 + y'^2)\right]^{5/2}}\left(\left[6\mu^4 + 5\mu^2(1 + \mu^2)y'^2 + 2(1 - \mu^2 + \mu^4)y'^4 - (1 + \mu^2)y'^6\right]\right.$$

$$\left. - y'^2(1 + y'^2)\left[2\mu^2 + (1 + \mu^2)y'^2\right]\left\{1 + \frac{E\left(\frac{\pi}{2},k\right) + \mu^2K\left(\frac{\pi}{2},k\right)}{y'^2\left[K\left(\frac{\pi}{2},k\right) - E\left(\frac{\pi}{2},k\right)\right] - E\left(\frac{\pi}{2},k\right)}\right\}\right) \geqq 0 \tag{43}$$

For $y(x) > 0$ and $C_5 = 0$, the slope $y'(x)$ is constant (eq. (41)). Thus, the bodies are cones with elliptical cross sections.

The complete solution to the problem is obtained in the following way: The integral constraint (given surface area) is (eq. (14))

$$\frac{\mu S}{2} = \int_0^{x_1} 2y\sqrt{1 + y'^2}E\left(\frac{\pi}{2}, k\right)dx$$

Since $y'(x) = \text{constant}$, $y = y'x$, so that

$$\frac{\mu S}{2} = 2y'\sqrt{1 + y'^2}E\left(\frac{\pi}{2}, k\right)\int_0^{x_1} x\, dx = y'\sqrt{1 + y'^2}E\left(\frac{\pi}{2}, k\right)x_1{}^2$$

However, $x_1 \equiv \dfrac{y(x_1)}{y'}$, and

$$\frac{\mu S}{2} = \frac{\sqrt{1 + y'^2}}{y'}\, E\left(\frac{\pi}{2}, k\right)\left[y(x_1)\right]^2$$

or

$$\frac{\mu S}{2\left[y(x_1)\right]^2} = \frac{\sqrt{1 + y'^2}}{y'}\, E\left(\frac{\pi}{2}, k\right) \tag{44}$$

Equation (44) may be solved for y' with given $y(x_1)$ and surface area. Further,

$$x_1 = \frac{y(x_1)}{y'} = l$$

may be evaluated.

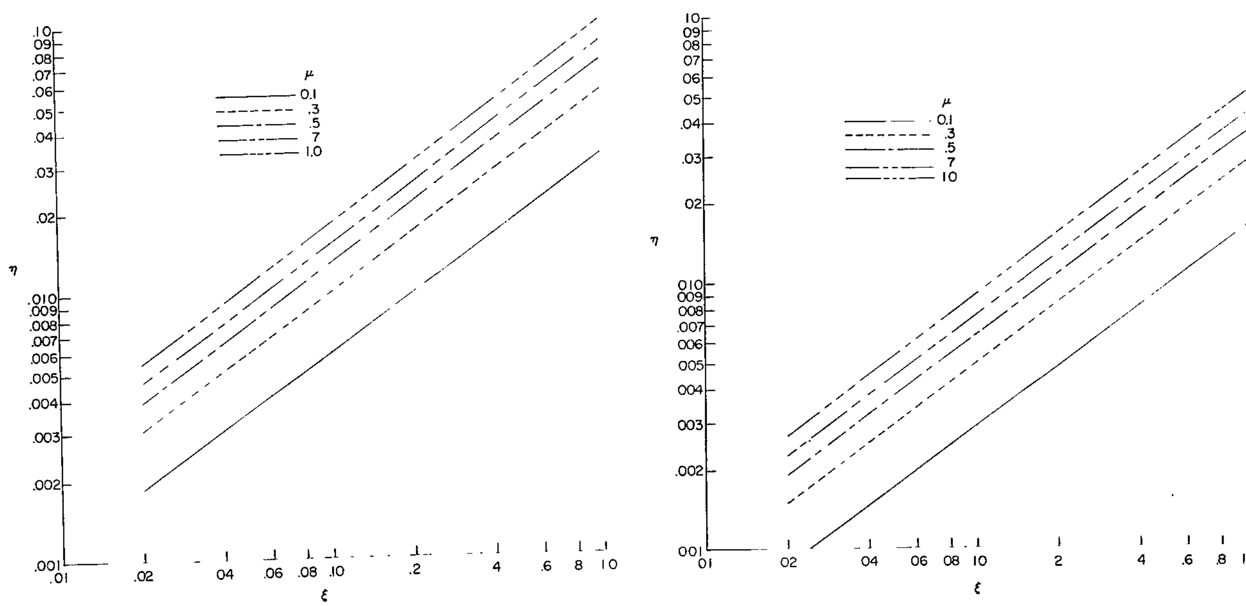
Finally, the value of y' from equation (44) may be substituted into equation (41) to evaluate λ. Of course, λ and y' must satisfy the Legendre condition, inequality (43), for an admissible solution.

RESULTS OF COMPUTATIONS

The problem of determining minimum pressure-drag body shapes with elliptical cross sections has been formulated, the solutions have been discussed qualitatively, and procedures have been presented to aid numerical solutions. The remainder of the present report will be a presentation of the results of computations for the conditions of given base height and length and given length and volume.

To assess the effect of body-section ellipticity on the geometry and pressure drag of minimum drag shapes, calculations were made for bodies with given length and base height and given length and volume. For convenience, the calculations were made for fineness ratios $\left(n \equiv \frac{\sqrt{\mu}\,l}{2y(l)}\right)$ of 5 and 10. Furthermore, body coordinates were made nondimensional by dividing by the body length l $\left(\text{that is,}\ \xi = \frac{x}{l},\ \text{and}\ \eta = \frac{y}{l}\right)$. The calculations were made for values of the ellipticity parameter μ ranging from 0.1 to 1.0.

Body shapes for given base height and length were calculated by programing equations (19) and (21) for a digital computer. Figures 4(a) and 4(b) are logarithmic plots of the bodies with fineness ratios of 5 and 10, respectively, and are presented to indicate the degree of validity of the approximation $\eta = m\xi^{p}$, where m and p are constants. This approximation plots as a straight line on a logarithmic scale, and the amount of deviation of the body shape curve from a straight line is a measure of the inaccuracy of the approximation. The results presented in figure 4 indicate that the approximation is exact. ~~except for $\mu = 0.1$~~. The value of p is about 0.75 for fineness ratios of 5 and 10, and is relatively insensitive to the parameter μ.

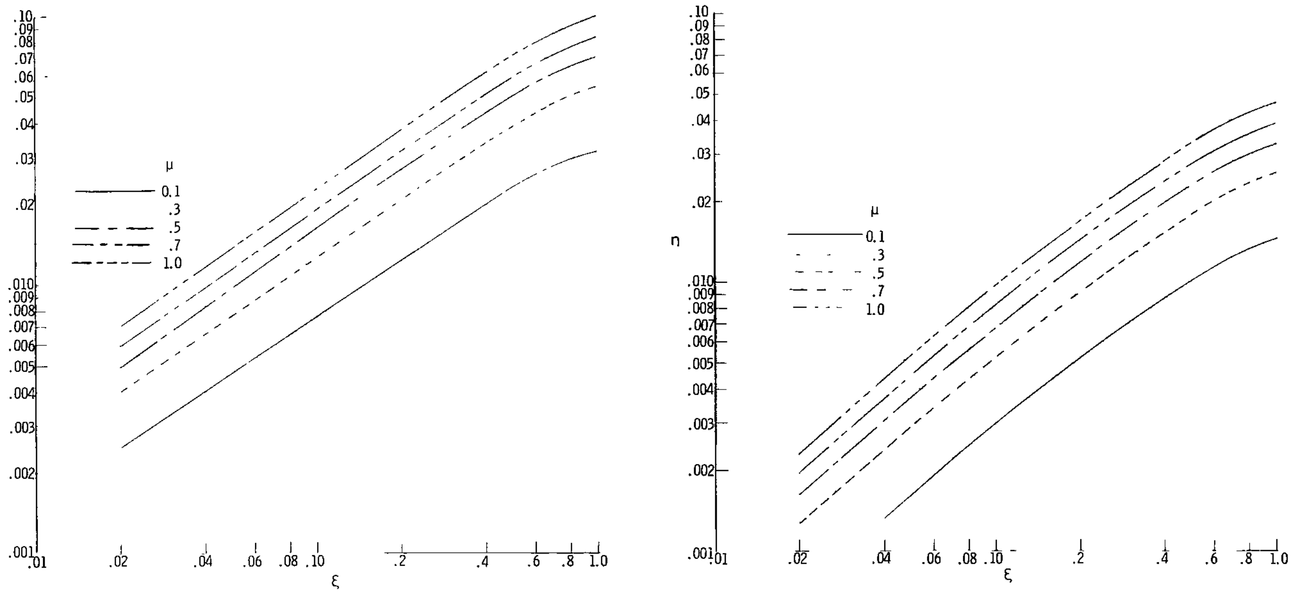


(a) Fineness ratio, 5.

(b) Fineness ratio, 10.

Figure 4.- Minimum drag body shapes for given length and base height.

The minimum drag body shapes for given length and volume were computed by the method of steepest descent although some body shapes were computed for checking purposes by the procedure outlined in the section "Specific Solutions." The results of the computations are presented in figures 5(a) and 5(b) for bodies with fineness ratios of 5 and 10, or l^3/V of 63.71 and 297.05, respectively. "The logarithmic plots in figure 5 show that the body-shape curves may not be approximated by a straight line except near the nose. Therefore, the shapes cannot be approximated by $\eta = m\xi^p$."



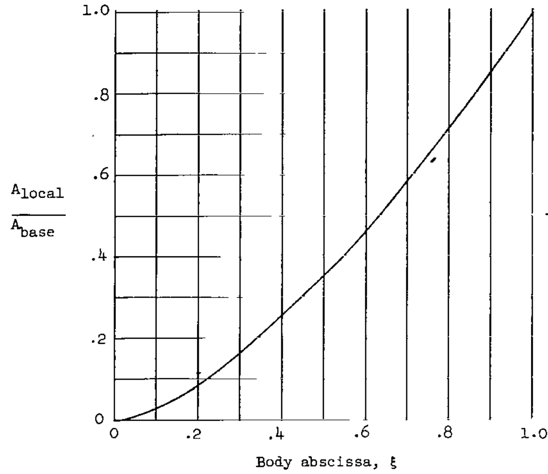
(a) Fineness ratio, 5.

(b) Fineness ratio, 10.

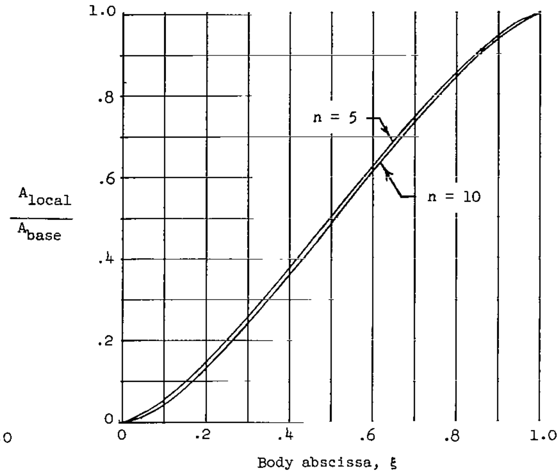
Figure 5.- Minimum drag body shapes for given length and volume.

In figure 6, the distribution of the cross-section area ratio $\frac{A_{local}}{A_{base}}$ along the body length is presented for the minimum drag body shapes with given length and base height and given length and volume. It should be noted that the area distribution does not depend on the ellipticity parameter μ in the interval $0.1 \leqq \mu \leqq 1.0$. Furthermore, the area distributions are valid for fineness ratios of 5 and 10. Therefore, for the given conditions, the range of the ellipticity, and the values of the fineness ratio, it would be sufficient to calculate only the minimum-drag body shape for one fineness ratio and an ellipticity of unity (that is, a body of revolution). The shapes for other ellipticities and fineness ratios could then be obtained from the resulting area distribution.

The effect of the ellipticity parameter on the pressure drag coefficient of the minimum drag bodies is shown in figure 7. For specified length and base

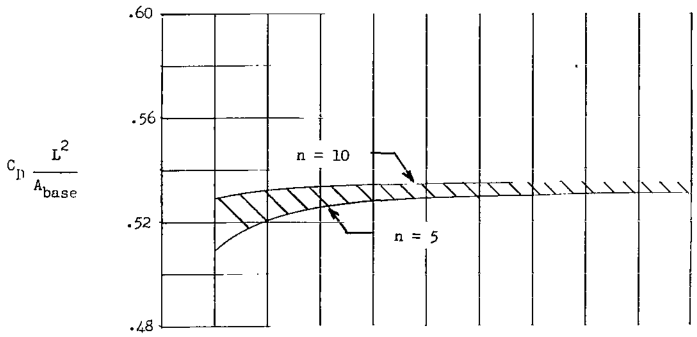


(a) Given length and base height.

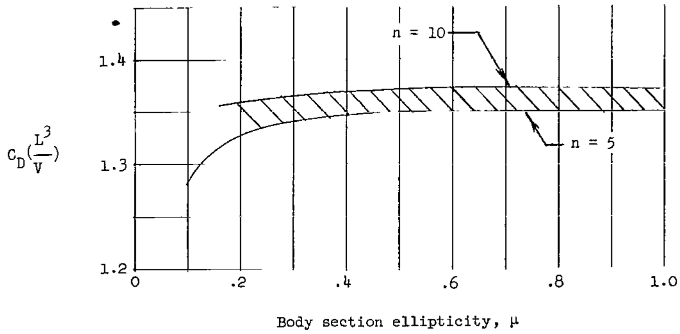


(b) Given length and volume.

Figure 6.- Area distribution of minimum drag bodies with fineness ratios of 5 and 10. Ellipticity, $0.1 \leqq \mu \leqq 1.0$. $\xi = x/l$.



(a) Given length and base height.



(b) Given length and volume.

Figure 7.- Pressure drag coefficient factor of minimum drag body shapes. Fineness ratios of 5 and 10.

height, the drag coefficients have been multiplied by the nondimensional ratio l^2/A_b; and for specified length and volume, the drag coefficients have been multiplied by the nondimensional ratio l^3/V. The drag-coefficient parameters, resulting from the use of these ratios, are independent of the fineness ratio, yet the trend of the variation with ellipticity is representative of the trend for the drag coefficient. The drag-coefficient parameter for the minimum drag body shapes for specified length and base height increases by about 150 percent

Page 19: The last three sentences of the paragraph continued at the top of this page should be changed as follows: "The drag-coefficient parameter for the minimum drag body shapes for specified length and base height and specified length and volume tends to decrease (although slightly) as the body sections become more elliptic (fig. 7). This result agrees with the theoretical and experimental results of Jorgensen (ref. 2) for elliptic cones."

CONCLUDING REMARKS

A theoretical study was made to investigate the problem of determining the shapes of bodies with cross-sectional ellipticity having minimum pressure drag at hypersonic speeds and zero lift. The results obtained include bodies with conditions of given length and base height, given length and volume, given base height and volume, and given base height and surface area. Much qualitative information about the body shapes was obtained without arriving at a complete solution to the problem. Although closed-form analytical solutions were not obtained for each set of given conditions, numerical solutions may be obtained with an iterative procedure.

The second paragraph under CONCLUDING REMARKS should read: "Numerical computations for bodies with given length and base height and given length and volume show that the longitudinal area distributions are relatively insensitive to the ellipticity parameters for fineness ratios of 5 and 10. In addition, the drag-coefficient parameters tended to decrease as the bodies became more elliptic. The latter result agrees with theoretical and experimental results for elliptic cones."

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., May 14, 1964.

APPENDIX A

METHOD OF STEEPEST DESCENT

The method of steepest descent has proven to be a powerful numerical computing tool for the optimization of a controlled plant when the quantity to be optimized is a function of the final values of the dependent variables. Problems in aerodynamics are not usually thought of in terms of control theory and controlled plants. However, in the analysis of the problem of determining minimum drag shapes, the method of steepest descent was recognized as a useful computational tool. In the following paragraphs, an exposition of the application of the method of steepest descent to a controlled plant having three variables is presented. The necessary reformulation of the minimum drag body problem is given so that the method of steepest descent can be used.

Let the set of ordinary differential equations of a system be

$$\frac{d\vec{x}}{dt} = \dot{\vec{x}} = \vec{f}\left(\vec{x}(t), u(t), t\right) \tag{A1}$$

where the components of the vector $\vec{x}(t)$ are the dependent state variables, the $\vec{f}$ is a given vector-valued function, $u(t)$ is the control variable, and t is the independent variable of the problem. (A dot above a quantity indicates differentiation with respect to the independent variable.) It will be assumed that the initial condition vector $\vec{x}(0) = \vec{x}_o$, is given and that $u(t)$ will be "guessed" along the nominal path. Thus, the set of equations (A1) may be integrated from $t = t_o$ to $t = T$. (The subscript o denotes initial conditions.)

The linear equations (three dimensional for convenience) that describe small perturbations about a nominal path are

$$\delta\dot{\vec{x}} = \left[A\right]\delta\vec{x} + \vec{m}\delta u \tag{A2}$$

$$\delta\dot{\vec{x}} = \begin{bmatrix} \dfrac{d\delta x_1}{dt} \\[2ex] \dfrac{d\delta x_2}{dt} \\[2ex] \dfrac{d\delta x_3}{dt} \end{bmatrix}$$

$$\delta\vec{x} = \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}$$

where

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}$$

and

$$\vec{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{bmatrix}$$

The elements of the matrices $\begin{bmatrix} A \end{bmatrix}$ and $\vec{m}$ are evaluated along the nominal path.

The set of equations adjoint to (A2) is defined to be:

$$\dot{\vec{\lambda}} = -\begin{bmatrix} A \end{bmatrix}^* \vec{\lambda} \qquad \text{(A3)}$$

where $\begin{bmatrix} A \end{bmatrix}^*$ is the transpose of $\begin{bmatrix} A \end{bmatrix}$. The ith $\lambda(t)$ is the influence function corresponding to the ith state variable $x_i(t)$. From equations (A2) and (A3), the following equation may be obtained:

$$\frac{d}{dt}\left(\vec{\lambda} \cdot \delta\vec{x}\right) = \vec{\lambda} \cdot \vec{m}\delta u \qquad \text{(A4)}$$

integrating over the interval of the path $t = t_o$ to $t = T$ gives

$$\vec{\lambda}(T) \cdot \delta\vec{x}(T) = \int_{t_o}^{T} \delta u \vec{\lambda} \cdot \vec{m}\, dt \tag{A5}$$

It will be assumed that T is given and that one terminal constraint is given.

Let the quantity to be optimized be $\phi\big(x(T)\big)$, a function depending on the final values of the dependent variables (for example, the function may be the final value of x_1, that is, $\phi\big(x(T)\big) = x_1(T)$. Then let

$$\lambda_i^{\phi}(T) = \left.\frac{\partial\phi}{\partial x_i}\right|_{t=T} \tag{A6}$$

be a set of influence functions associated with ϕ. Obviously,

$$\vec{\lambda}^{\phi}(T) \cdot \delta\vec{x}(T) = \delta\phi\big(x(T)\big)$$

and, hence, equation (A5), the influence of initial conditions being neglected, becomes

$$\delta\phi = \int_{t_o}^{T} \delta u\left(\vec{\lambda}^{\phi} \cdot \vec{m}\right)dt \tag{A7}$$

Now, $\vec{\lambda}^{\phi}(t)$ is the influence function that indicates the effect that small changes in the control function $\delta u(t)$ will have on ϕ, the function to be optimized.

Suppose there is a constraint on $\vec{x}(t)$ which is

$$\psi\big(\vec{x}(T)\big) = 0 \tag{A8}$$

and construct, as previously, another set of influence functions[1] associated with ψ, or

$$\lambda_i^{\psi}(T) = \left.\frac{\partial\psi}{\partial x_i}\right|_{t=T} \tag{A9}$$

---

[1]The influence functions $\vec{\lambda}^{\phi}$ and $\vec{\lambda}^{\psi}$ are calculated by integrating equations (A3) from $t = T$ to $t = t_o$ and using equations (A6) and (A9), respectively, as initial conditions.

As in equation (A7), the influence of initial conditions being neglected,

$$\delta\psi = \int_{t_o}^{T} \delta u\left(\vec{\lambda}^{\psi} \cdot \vec{m}\right)dt \tag{A10}$$

Now, the expression for the change in the control vector that gives the largest changes of ϕ and ψ in the right direction is

$$\delta u(t) = K_{\phi}\left(\vec{\lambda}^{\phi} \cdot \vec{m}\right) + K_{\psi}\left(\vec{\lambda}^{\psi} \cdot \vec{m}\right) \tag{A11}$$

where K_{ϕ} and K_{ψ} are constants. Substituting (A11) into (A7) and (A10), two linear equations for the two unknown constants K_{ϕ} and K_{ψ} are obtained:

$$\delta\phi = K_{\phi}I_{\phi\phi} + K_{\psi}I_{\psi\phi}$$

$$\delta\psi = K_{\phi}I_{\psi\phi} + K_{\psi}I_{\psi\psi}$$

which yields

$$\begin{bmatrix} K_{\phi} \\ K_{\psi} \end{bmatrix} = \frac{\begin{bmatrix} I_{\psi\psi} & -I_{\phi\psi} \\ -I_{\phi\psi} & I_{\phi\phi} \end{bmatrix}\begin{bmatrix} \delta\phi \\ \delta\psi \end{bmatrix}}{\begin{vmatrix} I_{\phi\phi} & I_{\phi\psi} \\ I_{\phi\psi} & I_{\psi\psi} \end{vmatrix}} \tag{A12}$$

where

$$I_{\phi\phi} = \int_{t_o}^{T}\left(\vec{\lambda}^{\phi} \cdot \vec{m}\right)^2 dt$$

$$I_{\psi\psi} = \int_{t_o}^{T}\left(\vec{\lambda}^{\psi} \cdot \vec{m}\right)^2 dt$$

$$I_{\phi\psi} = \int_{t_o}^{T}\left(\vec{\lambda}^{\phi} \cdot \vec{m}\right)\left(\vec{\lambda}^{\psi} \cdot \vec{m}\right)dt$$

The three integrals $I_{\phi\phi}$, $I_{\psi\psi}$, and $I_{\phi\psi}$ are calculated simultaneously with the influence functions. Now, $\delta\phi$ and $\delta\psi$ are "asked for" changes in ϕ

and ψ. The changes are in a direction to optimize ϕ and to satisfy the constraint (A8). From equation (A11), $\delta u(t)$ is computed and a new control vector U is determined for the next computer run, as

$$U_{New} = U_{Old} + \delta u$$

The process is repeated several times. When the optimum path that satisfies the constraint (A8) is approached, the determinant in equations (A12) will tend toward zero. The problem may as well be stopped whenever the determinant decreases enough to be troublesome because the optimum path is essentially reached.

The method of steepest descent may be used to determine the body shape that minimizes the pressure drag (assuming Newtonian flow and zero angle of attack) where the length and volume are given. The cross section of the body is elliptical. Some useful definitions are given below.

D	pressure drag, lb
l	body length, ft
q_∞	free-stream dynamic pressure, lb/sq ft
V	volume of body, cu ft
x	distance along center line of body (zero at nose of body)
y	distance perpendicular to body center line (function of x)
$y(0,\mu)$	value of y for specified μ when $x = 0$
μ	body section ellipticity (ratio of minor axis to major axis)

The problem is made compatible with the method of steepest descent by the following change in notation.

Let the independent variable be

$$t = \frac{x}{l} \tag{A13}$$

Let the "control" variable be given by

$$u(t) = y'(x) \equiv \frac{dy}{dx} \tag{A14}$$

Let the dependent variables (components of the vector $\vec{x}(t)$) be

$$x_1 = \frac{1}{2l^2}\left[\frac{\mu D}{2\pi q_\infty} - \left(y(0,\mu)\right)^2\right] = \int_0^1 \frac{x_3 u^3 dt}{\sqrt{\left(1 + u^2\right)\left(\mu^2 + u^2\right)}} \tag{A15}$$

$$x_2 = \frac{\mu V}{\pi l^3} = \int_0^1 x_3^2 dt \tag{A16}$$

$$x_3 = \frac{y}{l} \tag{A17}$$

Then the set of differential equations which describe the problem is

$$\left.\begin{aligned} \dot{x}_1 &= \frac{x_3 u^3}{\sqrt{(1 + u^2)(\mu^2 + u^2)}} \\ \dot{x}_2 &= x_3^2 \\ \dot{x}_3 &= u \end{aligned}\right\} \tag{A18}$$

The function to be minimized is

$$\phi\left(\vec{x}(T)\right) = x_1(T) \tag{A19}$$

and the constraint equation is

$$\psi\left(\vec{x}(T)\right) = x_2(T) - \frac{\mu V}{\pi l^3} = 0 \tag{A20}$$

With the problem formulated in terms of equations (A18), (A19), and (A20), the method outlined is immediately applicable. Solutions to the problem using this method with a digital computer system have been obtained in less than 3 minutes.

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